The Multiple Discrete-Continuous Extreme Value (MDCEV) Model: Formulation and Applications

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ABSTRACT

Many consumer choice situations are characterized by the simultaneous demand for multiple alternatives that are imperfect substitutes for one another. A simple and parsimonious Multiple Discrete-Continuous Extreme Value (MDCEV) econometric approach to handle such multiple discreteness was formulated by Bhat (2005) within the broader Kuhn-Tucker (KT) multiple discrete-continuous economic consumer demand model of Wales and Woodland (1983). In this chapter, the focus is on presenting the basic MDCEV model structure, discussing its estimation and use in prediction, formulating extensions of the basic MDCEV structure, and presenting applications of the model. The paper examines several issues associated with the MDCEV model and other extant KT multiple discrete-continuous models. Specifically, the paper discusses the utility function form that enables clarity in the role of each parameter in the utility specification, presents identification considerations associated with both the utility functional form as well as the stochastic nature of the utility specification, extends the MDCEV model to the case of price variation across goods and to general error covariance structures, discusses the relationship between earlier KT-based multiple discrete-continuous models, and illustrates the many technical nuances and identification considerations of the multiple discrete-continuous model structure. Finally, we discuss the many applications of MDCEV model and its extensions in various fields.

Keywords: Discrete-continuous system, Multiple discreteness, Kuhn-Tucker demand systems, Mixed discrete choice, Random Utility Maximization.

1. INTRODUCTION

Several consumer demand choices related to travel and other decisions are characterized by the choice of multiple alternatives simultaneously, along with a continuous quantity dimension associated with the consumed alternatives. Examples of such choice situations include vehicle type holdings and usage, and activity type choice and duration of time investment of participation. In the former case, a household may hold a mix of different kinds of vehicle types (for example, a sedan, a minivan, and a pick-up) and use the vehicles in different ways based on the preferences of individual members, considerations of maintenance/running costs, and the need to satisfy different functional needs (such as being able to travel on weekend getaways as a family or to transport goods). In the case of activity type choice and duration, an individual may decide to participate in multiple kinds of recreational and social activities within a given time period (such as a day) to satisfy variety seeking desires. Of course, there are several other travel-related and other consumer demand situations characterized by the choice of multiple alternatives, including airline fleet mix and usage, carrier choice and transaction level, brand choice and purchase quantity for frequently purchased grocery items (such as cookies, ready-to-eat cereals, soft drinks, yoghurt, *etc.*), and stock selection and investment amounts.

There are many ways that multiple discrete situations, such as those discussed above, may be modeled. One approach is to use the traditional random utility-based (RUM) single discrete choice models by identifying all combinations or bundles of the "elemental" alternatives, and treating each bundle as a "composite" alternative (the term "single discrete choice" is used to refer to the case where a decision-maker chooses only one alternative from a set of alternatives). A problem with this approach, however, is that the number of composite alternatives explodes with the number of elemental alternatives. Specifically, if J is the number of elemental alternatives, the total number of composite alternatives is ($2^J - 1$). A second approach to analyze multiple discrete situations is to use the multivariate probit (logit) methods of Manchanda *et al.* (1999), Baltas (2004), Edwards and Allenby (2003), and Bhat and Srinivasan (2005). In these multivariate methods, the multiple discreteness is handled through statistical methods that generate correlation between univariate utility maximizing models for single discreteness. While interesting, this second approach is more of a statistical "stitching" of univariate models rather than being fundamentally derived from a rigorous underlying utility maximization model for multiple discreteness. The resulting multivariate models also do not collapse to the standard discrete choice models when all individuals choose one and only one alternative at each choice occasion. A third approach is the one proposed by Hendel (1999) and Dube (2004). These researchers consider the case of "multiple discreteness" in the purchase of multiple varieties within a particular product category as the result of a stream of expected (but unobserved to the analyst) future consumption decisions between successive shopping purchase occasions (see also Walsh, 1995). During each consumption occasion, the standard discrete choice framework of perfectly substitutable alternatives is invoked, so that only one product is consumed. Due to varying tastes across individual consumption occasions between the current shopping purchase and the next, consumers are observed to purchase a variety of goods at the current shopping occasion.

In all the three approaches discussed above to handle multiple discreteness, there is no recognition that individuals choose multiple alternatives to satisfy different functional or variety seeking needs (such as wanting to relax at home as well as participate in out-of-home recreation). Thus, the approaches fail to incorporate the diminishing marginal returns (*i.e.*, satiation) in participating in a single type of activity, which may be the fundamental driving force for individuals choosing to participate in multiple activity types.[1] Finally, in the approaches above, it is very cumbersome, even if conceptually feasible, to include a continuous choice into the model (for example, modeling the different activity purposes of participation as well as the duration of participation in each activity purpose).

Wales and Woodland (1983) proposed two alternative ways to handle situations of multiple discreteness based on satiation behavior within a behaviorally-consistent utility maximizing framework. Both approaches assume a direct utility function $U(\boldsymbol{x})$ that is assumed to be quasi-concave, increasing, and continuously differentiable with respect to the consumption quantity vector $\boldsymbol{x}$.[2] Consumers maximize the utility function subject to a linear budget constraint, which is binding in that all the available budget is invested in the consumption of the goods; that

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[1] The approach of Hendel and Dube can be viewed as a "vertical" variety-seeking model that may be appropriate for frequently consumed grocery items such as carbonated soft drinks, cereals, and cookies. However, in many other choice occasions, such as time allocation to different types of discretionary activities, the true decision process may be better characterized as "horizontal" variety-seeking, where the consumer selects an assortment of alternatives due to diminishing marginal returns for each alternative. That is, the alternatives represent inherently imperfect substitutes at the choice occasion.

[2] The assumption of a quasi-concave utility function is simply a manifestation of requiring the indifference curves to be convex to the origin (see Deaton and Muellbauer, 1980, page 30 for a rigorous definition of quasi-concavity). The assumption of an increasing utility function implies that $U(\boldsymbol{x}^1) > U(\boldsymbol{x}^0)$ if $\boldsymbol{x}^1 > \boldsymbol{x}^0$.

is, the budget constraint has an equality sign rather than a '$\leq$' sign. This binding nature of the budget constraint is the result of assuming an increasing utility function, and also implies that at least one good will be consumed. The difference in the two alternative approaches proposed by Wales and Woodland (1983) is in how stochasticity, non-negativity of consumption, and corner solutions (*i.e.*, zero consumption of some goods) are accommodated, as briefly discussed below (see Wales and Woodland, 1983 and Phaneuf *et al.*, 2000 for additional details).

The first approach, which Wales and Woodland label as the Amemiya-Tobin approach, is an extension of the classic microeconomic approach of adding normally distributed stochastic terms to the budget-constrained utility-maximizing share equations. In this approach, the direct utility function $U(\boldsymbol{x})$ itself is assumed to be deterministic by the analyst, and stochasticity is introduced post-utility maximization. The justification for the addition of such normally distributed stochastic terms to the deterministic utility-maximizing allocations is based on the notion that consumers make errors in the utility-maximizing process, or that there are measurement errors in the collection of share data, or that there are unknown factors (from the analyst's perspective) influencing actual consumed shares. However, the addition of normally distributed error terms to the share equations in no way restricts the shares to be positive and less than 1. The contribution of Wales and Woodland was to devise a stochastic formulation, based on the earlier work of Tobin (1958) and Amemiya (1974), that (a) respects the unit simplex range constraint for the shares, (b) accommodates the restriction that the shares sum to one, and (c) allows corner solutions in which one or more alternatives are not consumed. They achieve this by assuming that the observed shares for the (K-1) of the K alternatives follow a truncated multivariate normal distribution (note that since the shares across alternatives have to sum to one, there is a singularity generated in the K-variate covariance matrix of the K shares, which can be accommodated by dropping one alternative). However, an important limitation of the Amemiya-Tobin approach of Wales and Woodland is that it does not account for corner solutions in its underlying behavior structure. Rather, the constraint that the shares have to lie within the unit simplex is imposed by *ad hoc* statistical procedures of mapping the density outside the unit simplex to the boundary points of the unit simplex.

The second approach suggested by Wales and Woodland, which they label as the Kuhn-Tucker approach, is based on the Kuhn Tucker or KT (1951) first-order conditions for constrained random utility maximization (see Hanemann, 1978, who uses such an approach even before Wales and Woodland). Unlike the Amemiya-Tobin approach, the KT approach employs a more direct stochastic specification by assuming the utility function $U(\boldsymbol{x})$ to be random (from the analyst's perspective) over the population, and then derives the consumption vector for the random utility specification subject to the linear budget constraint by using the KT conditions for constrained optimization. Thus, the stochastic nature of the consumption vector in the KT approach is based fundamentally on the stochastic nature of the utility function. Consequently, the KT approach immediately satisfies all the restrictions of utility theory, and the stochastic KT first-order conditions provide the basis for deriving the probabilities for each possible combination of corner solutions (zero consumption) for some goods and interior solutions (strictly positive consumption) for other goods. The singularity imposed by the "adding-up" constraint is accommodated in the KT approach by employing the usual differencing approach with respect to one of the goods, so that there are only (K-1) interdependent stochastic first-order conditions.

Among the two approaches discussed above, the KT approach constitutes a more theoretically unified and behaviorally consistent framework for dealing with multiple discreteness consumption patterns. However, the KT approach did not receive much attention until relatively recently because the random utility distribution assumptions used by Wales and Woodland led to a complicated likelihood function that entails multi-dimensional integration. Kim *et al.* (2002) addressed this issue by using the Geweke-Hajivassiliou-Keane (or GHK) simulator to evaluate the multivariate normal integral appearing in the likelihood function in the KT approach. Also, different from Wales and Woodland, Kim *et al.* used a generalized variant of the well-known translated constant elasticity of substitution (CES) direct utility function (see Pollak and Wales, 1992; page 28) rather than the quadratic direct utility function used by Wales and Woodland. In any case, the Kim *et al.* approach, like the Wales and Woodland approach, is unnecessarily complicated because of the need to evaluate truncated multivariate normal integrals in the likelihood function. In contrast, Bhat (2005) introduced a simple and parsimonious econometric approach to handle multiple discreteness, also based on the generalized variant of the translated CES utility function but with a multiplicative log-extreme value error term. Bhat's model, labeled the multiple discrete-continuous extreme value (MDCEV) model, is analytically tractable in the probability expressions and is practical even for situations with a large number of discrete consumption alternatives. In fact, the MDCEV model represents the multinomial logit (MNL) form-equivalent for multiple discrete-continuous choice analysis and collapses exactly to the MNL in the case that each (and every) decision-maker chooses only one alternative.

Independent of the above works of Kim *et al.* and Bhat, there has been a stream of research in the environmental economics field (see Phaneuf *et al.*, 2000; von Haefen *et al.*, 2004; von Haefen, 2003; von Haefen, 2004; von Haefen and Phaneuf, 2005; Phaneuf and Smith, 2005) that has also used the KT approach to multiple discreteness. These studies use variants of the linear expenditure system (LES) as proposed by Hanemann (1978) and the translated CES for the utility functions, and use multiplicative log-extreme value errors. However, the error specification in the utility function is different from that in Bhat's MDCEV model, resulting in a different form for the likelihood function.

In this chapter, the focus is on presenting the basic MDCEV model structure, discussing its estimation and use in prediction, formulating extensions of the basic MDCEV structure, and presenting applications of the model. Accordingly, the rest of the chapter is structured as follows. The next section formulates a functional form for the utility specification that enables the isolation of the role of different parameters in the specification. This section also identifies empirical identification considerations in estimating the parameters in the utility specification. Section 3 discusses the stochastic form of the utility specification, the resulting general structure for the probability expressions, and associated identification considerations. Section 4 derives the MDCEV structure for the utility functional form used in the current paper, and extends this structure to more general error structure specifications. For presentation ease, Sections 2 through 4 consider the case of the absence of an outside good. In Section 5, we extend the discussions of the earlier sections to the case when an outside good is present. Section 6 provides an overview of empirical applications using the model. The final section concludes the paper.

2. FUNCTIONAL FORM OF UTILITY SPECIFICATION

We consider the following functional form for utility in this paper, based on a generalized variant of the translated CES utility function:

$$U(\boldsymbol{x}) = \sum_{k=1}^{K} \frac{\gamma_k}{\alpha_k} \psi_k \left\{ \left(\frac{x_k}{\gamma_k} + 1 \right)^{\alpha_k} - 1 \right\} \tag{1}$$

where $U(\boldsymbol{x})$ is a quasi-concave, increasing, and continuously differentiable function with respect to the consumption quantity (Kx1)-vector $\boldsymbol{x}$ ($x_k \geq 0$ for all k), and ψ_k, γ_k and α_k are parameters associated with good k. The function in Equation (1) is a valid utility function if $\psi_k > 0$ and $\alpha_k \leq 1$ for all k. Further, for presentation ease, we assume temporarily that there is no outside good, so that corner solutions (*i.e.*, zero consumptions) are allowed for all the goods k (this assumption is being made only to streamline the presentation and should not be construed as limiting in any way; the assumption is relaxed in a straightforward manner as discussed in Section 5). The possibility of corner solutions implies that the term γ_k, which is a translation parameter, should be greater than zero for all k.[3] The reader will note that there is an assumption of additive separability of preferences in the utility form of Equation (1), which immediately implies that none of the goods are *a priori* inferior and all the goods are strictly Hicksian substitutes (see Deaton and Muellbauer, 1980; page 139). Additionally, additive separability implies that the marginal utility with respect to any good is independent of the levels of all other goods.[4]

The form of the utility function in Equation (1) highlights the role of the various parameters ψ_k, γ_k and α_k, and explicitly indicates the inter-relationships between these parameters that relate to theoretical and empirical identification issues. The form also assumes weak complementarity (see Mäler, 1974), which implies that the consumer receives no utility from a non-essential good's attributes if s/he does not consume it (*i.e.*, a good and its quality attributes are weak complements, or $U_k = 0$ if $x_k = 0$, where U_k is the sub-utility function for the k*th* good). The reader will also note that the functional form proposed by Bhat (2008) in Equation (1) generalizes earlier forms used by Hanemann (1978), von Haefen *et al.* (2004), Herriges *et al.* (2004), Phaneuf *et al.* (2000) and Mohn and Hanemann (2005). Specifically, it should be noted that the utility form of Equation (1) collapses to the following linear expenditure system (LES) form when $\alpha_k \to 0 \; \forall \; k$:

[3] As illustrated in Kim *et al.* (2002) and Bhat (2005), the presence of the translation parameters makes the indifference curves strike the consumption axes at an angle (rather than being asymptotic to the consumption axes), thus allowing corner solutions.

[4] Some other studies assume the overall utility to be derived from the characteristics embodied in the goods, rather than using the goods as separate entities in the utility function. The reader is referred to Chan (2006) for an example of such a characteristics approach to utility. Also, as we discuss later, recent work by Vasquez and Hanemann (2008) relaxes the assumption of additive separability, but at a computational and interpretation cost.

$$U(\boldsymbol{x}) = \sum_{k=1}^{K} \gamma_k \psi_k \ln\left(\frac{x_k}{\gamma_k} + 1\right) \tag{2}$$

2.1 Role of Parameters in Utility Specification

2.1.1 Role of ψ_k

The role of ψ_k can be inferred by computing the marginal utility of consumption with respect to good *k*, which is:

$$\frac{\partial U(\boldsymbol{x})}{\partial x_k} = \psi_k \left(\frac{x_k}{\gamma_k} + 1\right)^{\alpha_k - 1} \tag{3}$$

It is obvious from above that ψ_k represents the baseline marginal utility, or the marginal utility at the point of zero consumption. Alternatively, the marginal rate of substitution between any two goods *k* and *l* at the point of zero consumption of both goods is $\frac{\psi_k}{\psi_l}$. This is the case regardless of the values of γ_k and α_k. For two goods *i* and *j* with same unit prices, a higher baseline marginal utility for good *i* relative to good *j* implies that an individual will increase overall utility more by consuming good *i* rather than *j* at the point of no consumption of any goods. That is, the consumer will be more likely to consume good *i* than good *j*. Thus, a higher baseline ψ_k implies less likelihood of a corner solution for good *k*.

2.1.2 Role of γ_k

An important role of the γ_k terms is to shift the position of the point at which the indifference curves are asymptotic to the axes from (0,0,0…,0) to $(-\gamma_1, -\gamma_2, -\gamma_3, \ldots, -\gamma_K)$, so that the indifference curves strike the positive orthant with a finite slope. This, combined with the consumption point corresponding to the location where the budget line is tangential to the indifference curve, results in the possibility of zero consumption of good *k*. To see this, consider two goods 1 and 2 with $\psi_1 = \psi_2 = 1$, $\alpha_1 = \alpha_2 = 0.5$, and $\gamma_2 = 1$. Figure 1 presents the profiles of the indifference curves in this two-dimensional space for various values of γ_1 ($\gamma_1 > 0$). To compare the profiles, the indifference curves are all drawn to go through the point (0,8). The reader will also note that all the indifference curve profiles strike the y-axis with the same slope. As can be observed from the figure, the positive values of γ_1 and γ_2 lead to indifference curves that cross the axes of the positive orthant, allowing for corner solutions. The indifference curve profiles are asymptotic to the x-axis at $y = -1$ (corresponding to the constant value of γ_2 = 1), while they are asymptotic to the y-axis at $x = -\gamma_1$.

Figure 1 also points to another role of the γ_k term as a satiation parameter. Specifically, the indifference curves get steeper in the positive orthant as the value of γ_1 increases, which implies a stronger preference (or lower satiation) for good 1 as γ_1 increases (with steeper indifference curve slopes, the consumer is willing to give up more of good 2 to obtain 1 unit of good 1). This point is particularly clear if we examine the profile of the sub-utility function for alternative k. Figure 2 plots the function for alternative k for $\alpha_k \to 0$ and ψ_k = 1, and for different values of γ_k. All of the curves have the same slope ψ_k = 1 at the origin point, because of the functional form used in this paper. However, the marginal utilities vary for the different curves at x_k > 0. Specifically, the higher the value of γ_k, the less is the satiation effect in the consumption of x_k.

2.1.3 Role of α_k

The express role of α_k is to reduce the marginal utility with increasing consumption of good k; that is, it represents a satiation parameter. When α_k = 1 for all k, this represents the case of absence of satiation effects or, equivalently, the case of constant marginal utility. The utility function in Equation (1) in such a situation collapses to $\sum_k \psi_k x_k$, which represents the perfect substitutes case as proposed by Deaton and Muellbauer (1980) and applied in Hanemann (1984), Chiang (1991), Chintagunta (1993), and Arora *et al.* (1998), among others. Intuitively, when there is no satiation and the unit good prices are all the same, the consumer will invest all expenditure on the single good with the highest baseline (and constant) marginal utility (*i.e.*, the highest ψ_k value). This is the case of single discreteness.[5] As α_k moves downward from the value of 1, the satiation effect for good *k* increases. When $\alpha_k \to 0$, the utility function collapses to the form in Equation (2), as discussed earlier. α_k can also take negative values and, when $\alpha_k \to -\infty$, this implies immediate and full satiation. Figure 3 plots the utility function for alternative *k* for γ_k = 1 and ψ_k = 1, and for different values of α_k. Again, all of the curves have the same slope ψ_k = 1 at the origin point, and accommodate different levels of satiation through different values of α_k for any given γ_k value.

2.2 Empirical Identification Issues Associated with Utility Form

The discussion in the previous section indicates that ψ_k reflects the baseline marginal utility, which controls whether or not a good is selected for positive consumption (or the extensive margin of choice). The role of γ_k is to enable corner solutions, though it also governs the level of satiation. The purpose of α_k is solely to allow satiation. Thus, for a given extensive margin of choice of good *k*, γ_k and α_k influence the quantity of good *k* consumed (or the intensive margin of choice) through their impact on satiation effects. The precise functional mechanism through which γ_k and α_k impact satiation are, however, different; γ_k controls satiation by translating consumption quantity, while α_k controls satiation by exponentiating consumption quantity. Clearly, both these effects operate in different ways, and different combinations of their values lead to different satiation profiles. However, empirically speaking, it is very difficult to disentangle the two effects separately, which leads to serious empirical identification problems and estimation breakdowns when one attempts to estimate both γ_k and α_k parameters for each good. In fact, for a given ψ_k value, it is possible to closely approximate a sub-utility function profile based on a combination of γ_k and α_k values with a sub-utility function based solely on γ_k or α_k values. In actual application, it would behoove the analyst to estimate models based on

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[5] If there is price variation across goods, one needs to take the derivative of the utility function with respect to expenditures (e_k) on the goods. In the case that α_k= 1 for all *k*, $U = \sum_k \psi_k(e_k / p_k)$, where ψ_k is the unit price of good *k*. Then $\partial U / \partial e_k = \psi_k / p_k$. In this situation, the consumer will invest all expenditures on the single good with the highest price-normalized marginal (and constant) utility ψ_k / p_k.

both the α_k-profile and the γ_k-profile, and choose a specification that provides a better statistical fit.[6]

3. STOCHASTIC FORM OF UTILITY FUNCTION

The KT approach employs a direct stochastic specification by assuming the utility function $U(\boldsymbol{x})$ to be random over the population. In all recent applications of the KT approach for multiple discreteness, a multiplicative random element is introduced to the baseline marginal utility of each good as follows:

$$\psi(z_k, \varepsilon_k) = \psi(z_k) \cdot e^{\varepsilon_k}, \tag{4}$$

where z_k is a set of attributes characterizing alternative *k* and the decision maker, and ε_k captures idiosyncratic (unobserved) characteristics that impact the baseline utility for good *j*. The exponential form for the introduction of the random term guarantees the positivity of the baseline utility as long as $\psi(z_k) > 0$. To ensure this latter condition, $\psi(z_k)$ is further parameterized as $\exp(\beta' z_k)$, which then leads to the following form for the baseline random utility associated with good *k*:

$$\psi(z_k, \varepsilon_k) = \exp(\beta' z_k + \varepsilon_k). \tag{5}$$

The z_k vector in the above equation includes a constant term. The overall random utility function of Equation (1) then takes the following form:

$$U(\boldsymbol{x}) = \sum_k \frac{\gamma_k}{\alpha_k} \left[\exp(\beta' z_k + \varepsilon_k)\right] \cdot \left\{ \left(\frac{x_k}{\gamma_k} + 1 \right)^{\alpha_k} - 1 \right\} \tag{6}$$

From the analyst's perspective, the individual is maximizing random utility subject to the binding linear budget constraint that $\sum_{k=1}^{K} e_k = E$, where *E* is total expenditure or income (or some other appropriately defined total budget quantity), $e_k = p_k x_k$, and p_k is the unit price of good *k*.

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[6] Alternatively, the analyst can stick with one functional form *a priori*, but experiment with various fixed values of α_k for the γ_k-profile and γ_k for the α_k-profile.

3.1 Optimal Expenditure Allocations

The analyst can solve for the optimal expenditure allocations by forming the Lagrangian and applying the Kuhn-Tucker (KT) conditions.[7] The Lagrangian function for the problem is:

$$\mathrm{L} = \sum_k \frac{\gamma_k}{\alpha_k}\left[\exp(\beta' z_k + \varepsilon_k)\right]\left\{\left(\frac{e_k}{\gamma_k p_k}+1\right)^{\alpha_k} - 1\right\} - \lambda\left[\sum_{k=1}^{K} e_k - E\right], \tag{7}$$

where λ is the Lagrangian multiplier associated with the expenditure constraint (that is, it can be viewed as the marginal utility of total expenditure or income). The KT first-order conditions for the optimal expenditure allocations (the e_k^* values) are given by:

$$\left[\frac{\exp(\beta' z_k + \varepsilon_k)}{p_k}\right]\left(\frac{e_k^*}{\gamma_k p_k}+1\right)^{\alpha_k - 1} - \lambda = 0\text{, if } e_k^* > 0\text{, } k = 1, 2,\dots, K \tag{8}$$

$$\left[\frac{\exp(\beta' z_k + \varepsilon_k)}{p_k}\right]\left(\frac{e_k^*}{\gamma_k p_k}+1\right)^{\alpha_k - 1} - \lambda < 0\text{, if } e_k^* = 0\text{, } k = 1, 2,\dots, K$$

The optimal demand satisfies the conditions in Equation (8) plus the budget constraint $\sum_{k=1}^{K} e_k^* = E$. The budget constraint implies that only K-1 of the e_k^* values need to be estimated, since the quantity consumed of any one good is automatically determined from the quantity consumed of all the other goods. To accommodate this constraint, designate activity purpose 1 as a purpose to which the individual allocates some non-zero amount of consumption (note that the individual should participate in at least one of the K purposes, given that $E > 0$). For the first good, the KT condition may then be written as:

$$\lambda = \frac{\exp(\beta' z_1 + \varepsilon_1)}{p_1}\left(\frac{e_1^*}{\gamma_1 p_1}+1\right)^{\alpha_1 - 1} \tag{9}$$

Substituting for λ from above into Equation (8) for the other activity purposes ($k = 2,\dots, K$), and taking logarithms, we can rewrite the KT conditions as:

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[7] For reasons that will become clear later, we solve for the optimal expenditure allocations e_k for each good, not the consumption amounts x_k of each good. This is different from earlier studies that focus on the consumption of goods.

$$V_k + \varepsilon_k = V_1 + \varepsilon_1 \text{ if } e_k^* > 0 \;\; (k = 2, 3, \ldots, K)$$

$$V_k + \varepsilon_k < V_1 + \varepsilon_1 \text{ if } e_k^* = 0 \;\; (k = 2, 3, \ldots, K), \text{ where} \tag{10}$$

$$V_k = \beta' z_k + (\alpha_k - 1)\ln\left(\frac{e_k^*}{\gamma_k p_k} + 1\right) - \ln p_k \;\; (k = 1, 2, 3, \ldots, K).$$

Also, note that, in Equation (10), a constant cannot be identified in the $\beta' z_k$ term for one of the K alternatives (because only the difference in the V_k from V_1 matters). Similarly, individual-specific variables are introduced in the V_k's for (K-1) alternatives, with the remaining alternative serving as the base.[8]

3.2 General Econometric Model Structure and Identification

To complete the model structure, the analyst needs to specify the error structure. In the general case, let the joint probability density function of the ε_k terms be $f(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_K)$. Then, the probability that the individual allocates expenditure to the first M of the K goods is:

$$P(e_1^*, e_2^*, e_3^*, \ldots, e_M^*, 0, 0, \ldots, 0) = |J| \int_{\varepsilon_1=-\infty}^{+\infty} \int_{\varepsilon_{M+1}=-\infty}^{V_1 - V_{M+1} + \varepsilon_1} \int_{\varepsilon_{M+2}=-\infty}^{V_1 - V_{M+2} + \varepsilon_1} \cdots \int_{\varepsilon_{K-1}=-\infty}^{V_1 - V_{K-1} + \varepsilon_1} \int_{\varepsilon_K=-\infty}^{V_1 - V_K + \varepsilon_1}$$
$$f(\varepsilon_1, V_1 - V_2 + \varepsilon_1, V_1 - V_3 + \varepsilon_1, \ldots, V_1 - V_M + \varepsilon_1, \varepsilon_{M+1}, \varepsilon_{M+2}, \ldots, \varepsilon_{K-1}, \varepsilon_K) \tag{11}$$
$$d\varepsilon_K d\varepsilon_{K-1} \ldots d\varepsilon_{M+2} d\varepsilon_{M+1} d\varepsilon_1 \,,$$

where J is the Jacobian whose elements are given by (see Bhat, 2005):

$$J_{ih} = \frac{\partial[V_1 - V_{i+1} + \varepsilon_1]}{\partial e_{h+1}^*} = \frac{\partial[V_1 - V_{i+1}]}{\partial e_{h+1}^*}; \; i, h = 1, 2, \ldots, M - 1. \tag{12}$$

The probability expression in Equation (11) is a (K-M+1)-dimensional integral. The expression for the probability of all goods being consumed is one-dimensional, while the expression for the probability of only the first good being consumed is K-dimensional. The dimensionality of the

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[8] These identification conditions are similar to those in the standard discrete choice model, though the origin of the conditions is different between standard discrete choice models and the multiple discrete-continuous models. In standard discrete choice models, individuals choose the alternative with highest utility, so that all that matters is relative utility. In multiple discrete-continuous models, the origin of these conditions is the adding up (or budget) constraint associated with the quantity of consumption of each good that leads to the KT first order conditions of Equation (10).

integral can be reduced by one by noticing that the KT conditions can also be written in a differenced form. To do so, define $\tilde{\varepsilon}_{k1} = \varepsilon_k - \varepsilon_1$, and let the implied multivariate distribution of the error differences be $g(\tilde{\varepsilon}_{21}, \tilde{\varepsilon}_{31}, ..., \tilde{\varepsilon}_{K1})$. Then, Equation (11) may be written in the equivalent (K-M)-integral form shown below:

$$P(e_1^*, e_2^*, e_3^*, ..., e_M^*, 0, 0, ..., 0) = |J| \int_{\tilde{\varepsilon}_{M+1,1}=-\infty}^{V_1 - V_{M+1}} \int_{\tilde{\varepsilon}_{M+2,1}=-\infty}^{V_1 - V_{M+2}} \cdots \int_{\tilde{\varepsilon}_{K-1,1}=-\infty}^{V_1 - V_{K-1}} \int_{\tilde{\varepsilon}_{K,1}=-\infty}^{V_1 - V_K}$$
$$g(V_1 - V_2, V_1 - V_3, ..., V_1 - V_M, \tilde{\varepsilon}_{M+1,1}, \tilde{\varepsilon}_{M+2,1}, ..., \tilde{\varepsilon}_{K,1}) d\tilde{\varepsilon}_{K,1} d\tilde{\varepsilon}_{K-1,1} ... d\tilde{\varepsilon}_{M+1,1} \quad (13)$$

The equation above indicates that the probability expression for the observed optimal expenditure pattern of goods is completely characterized by the (K-1) error terms in difference form. Thus, all that is estimable is the (K-1)x(K-1) covariance matrix of the error differences. In other words, it is not possible to estimate a full covariance matrix for the original error terms $(\varepsilon_1, \varepsilon_2, ..., \varepsilon_K)$ because there are infinite possible densities for f(.) that can map into the same g(.) density for the error differences (see Train, 2003, page 27, for a similar situation in the context of standard discrete choice models). There are many possible ways to normalize f(.) to account for this situation. For example, one can assume an identity covariance matrix for f(.), which automatically accommodates the normalization that is needed. Alternatively, one can estimate g(.) without reference to f(.).

In the general case when the unit prices p_k vary across goods, it is possible to estimate $K*(K-1)/2$ parameters of the full covariance matrix of the error differences, as just discussed (though the analyst might want to impose constraints on this full covariance matrix for ease in interpretation and stability in estimation). However, when the unit prices are not different among the goods, an additional scaling restriction needs to be imposed. To see this, consider the case of independent and identically distributed error terms for the ε_k terms, which leads to a (K-1)x(K-1) covariance matrix for $\tilde{\varepsilon}_{k1}$ (k = 2,3,…,K) with diagonal elements equal to twice the value of scale parameter of the ε_k terms and off-diagonal elements equal to the scale parameter of the ε_k terms. Let the unit prices of all goods be the same (see Bhat, 2005; Bhat and Sen, 2006; Bhat *et al.*, 2006 and Bhat *et al.*, 2009 for examples where the weights or prices on the goods in the budget constraint are equal). Consider the utility function in Equation (6) and another utility function as given below:

$$\widetilde{U} = \sum_k \frac{\gamma_k}{\alpha_k^*} [\exp\{\sigma \times (\beta' z_k + \varepsilon_k)\}] \cdot \left\{ \left(\frac{x_k}{\gamma_k} + 1 \right)^{\alpha_k^*} - 1 \right\} \tag{14}$$

The scale of the error terms in the utility function in the above expression is σ times the scale of the error terms in Equation (6). Let $\alpha_k^* = \sigma(\alpha_k - 1) + 1$, where α_k is the satiation parameter in the original Equation (6).[9] The KT conditions for optimal expenditure for this modified utility function can be shown to be:

$$V_k^* + \sigma\varepsilon_k = V_1^* + \sigma\varepsilon_1 \text{ if } e_k^* > 0 \ \ (k = 2, 3, \ldots, K)$$

$$V_k^* + \sigma\varepsilon_k < V_1^* + \sigma\varepsilon_1 \text{ if } e_k^* = 0 \ \ (k = 2, 3, \ldots, K)\text{, where} \tag{15}$$

$$\begin{aligned} V_k^* &= \sigma\beta' z_k + (\alpha_k^* - 1) \ln\left(\frac{e_k^*}{\gamma_k p_k} + 1 \right) - \ln p_k \quad (k = 1, 2, 3, \ldots, K) \\ &= \sigma\beta' z_k + \sigma(\alpha_k - 1) \ln\left(\frac{e_k^*}{\gamma_k p_k} + 1 \right) - \ln p_k \ \ (k = 1, 2, 3, \ldots, K). \end{aligned}$$

If the unit prices are not all the same (*i.e.*, the unit prices of at least two of the K goods are different), the KT conditions above are different from the KT conditions in Equation (10).

4. SPECIFIC MODEL STRUCTURES

4.1 The MDCEV Model Structure

Following Bhat (2005, 2008), consider an extreme value distribution for ε_k and assume that ε_k is independent of z_k (k = 1, 2, …, K) . The ε_k's are also assumed to be independently distributed across alternatives with a scale parameter of σ (σ can be normalized to one if there is no variation in unit prices across goods). Let V_k be defined as follows:

$$\begin{aligned} V_k &= \beta' z_k + (\alpha_k - 1) \ln\left(\frac{e_k^*}{p_k} + 1 \right) - \ln p_k \ \ (k = 1, 2, 3, \ldots, K)\text{, when the } \alpha\text{-profile is used, and} \\ V_k &= \beta' z_k - \ln\left(\frac{e_k^*}{\gamma_k p_k} + 1 \right) - \ln p_k \ (k = 1, 2, 3, \ldots, K)\text{, when the } \gamma\text{-profile is used.} \end{aligned} \tag{16}$$

[9] Note that α_k^* is less than or equal to 1 by definition, because α_k is less than or equal to 1 and the scale σ should be non-negative.

As discussed earlier, it is generally not possible to estimate the V_k form in Equation (10), because the α_k terms and γ_k terms serve a similar satiation role.

From Equation (11), the probability that the individual allocates expenditure to the first M of the K goods ($M \geq 1$) is:

$$P\left(e_1^*, e_2^*, e_3^*, ..., e_M^*, 0, 0, ..., 0\right)$$
$$= |J| \int_{\varepsilon_1=-\infty}^{\varepsilon_1=+\infty} \left\{ \left(\prod_{i=2}^{M} \frac{1}{\sigma} \lambda \left[\frac{V_1 - V_i + \varepsilon_1}{\sigma} \right] \right) \right\} \times \left\{ \prod_{s=M+1}^{K} \Lambda \left[\frac{V_1 - V_s + \varepsilon_1}{\sigma} \right] \right\} \frac{1}{\sigma} \lambda \left(\frac{\varepsilon_1}{\sigma} \right) d\varepsilon_1 , \tag{17}$$

where λ is the standard extreme value density function and Λ is the standard extreme value cumulative distribution function. The expression in Equation (17) simplifies to a remarkably simple and elegant closed-form expression. Bhat derived the form of the Jacobian for the case of equal unit prices across goods, which however can be extended in a simple fashion to accommodate the more general case of different unit prices. The resulting form for the determinant of the Jacobian has a compact structure given by:

$$|J| = \left(\prod_{i=1}^{M} c_i \right) \left(\sum_{i=1}^{M} \frac{1}{c_i} \right), \text{ where } c_i = \left(\frac{1-\alpha_i}{e_i^* + \gamma_i p_i} \right). \tag{18}$$ [10]

The integration in Equation (17) also collapses to a closed form expression providing the following overall expression:

$$P\left(e_1^*, e_2^*, e_3^*, ..., e_M^*, 0, 0, ..., 0\right)$$
$$= \frac{1}{\sigma^{M-1}} \left[\prod_{i=1}^{M} c_i \right] \left[\sum_{i=1}^{M} \frac{1}{c_i} \right] \left[\frac{\prod_{i=1}^{M} e^{V_i/\sigma}}{\left(\sum_{k=1}^{K} e^{V_k/\sigma} \right)^M} \right] (M-1)! \tag{19}$$

In the case when $M = 1$ (*i.e.*, only one alternative is chosen), there are no satiation effects (α_k =1 for all k) and the Jacobian term drops out (that is, the continuous component drops out, because all expenditure is allocated to good 1). Then, the model in Equation (19) collapses to the standard

[10] It is important to note that this compact Jacobian form is independent of the assumptions regarding the density and correlation structure of the error terms.

MNL model. Thus, the MDCEV model is a multiple discrete-continuous extension of the standard MNL model.[11]

The expression for the probability of the consumption pattern of the goods (rather than the expenditure pattern) can be derived to be:

$$P\left(x_1^*, x_2^*, x_3^*, ..., x_M^*, 0, 0, ..., 0\right)$$
$$= \frac{1}{p_1} \cdot \frac{1}{\sigma^{M-1}} \left[\prod_{i=1}^{M} f_i\right]\left[\sum_{i=1}^{M} \frac{p_i}{f_i}\right]\left[\frac{\prod_{i=1}^{M} e^{V_i/\sigma}}{\left(\sum_{k=1}^{K} e^{V_k/\sigma}\right)^M}\right](M-1)!, \tag{20}$$

where V_k is as defined earlier (see Equation 16) and $f_i = \left(\frac{1-\alpha_i}{x_i^* + \gamma_i}\right)$. The expression in Equation (20) is, however, not independent of the good that is used as the first one (see the $1/p_1$ term in front). In particular, different probabilities of the same consumption pattern arise depending on the good that is labeled as the first good (note that any good that is consumed may be designated as the first good). In terms of the likelihood function, the $1/p_1$ term can be ignored, since it is simply a constant in each individual's likelihood function. Thus, the same parameter estimates will result independent of the good designated as the first good for each individual, but it is still awkward to have different probability values for the same consumption pattern. This is particularly the case because different log-likelihood values at convergence will be obtained for different designations of the first good. Thus, the preferred approach is to use the probability expression for expenditure allocations, which will provide the same probability for a given expenditure pattern regardless of the good labeled as the first good. However, in the case that the first good is an outside numeraire good that is always consumed (see Section 5), then $p_1 = 1$ and one can use the consumption pattern probability expression or the expenditure allocation probability expression.

[11] Note that when $\alpha_k = 1$ for all k, $V_k = \beta' z_k - \ln p_k$. Even if $M = 1$, when Equation (19) collapses to the MNL form, the scale σ is estimable as long as the utility takes the functional form $V_k = \beta' z_k - \ln p_k$ and there is price variation across goods. This is because the scale is the inverse of the coefficient on the $\ln p_k$ term (see Hanemann, 1984).

4.2 The Multiple Discrete-Continuous Generalized Extreme-Value (MDCGEV) Model Structure

Thus far, we have assumed that the ε_k terms are independently and identically extreme value distributed across alternatives *k*. The analyst can extend the model to allow correlation across alternatives using a generalized extreme value (GEV) error structure. The remarkable advantage of the GEV structure is that it continues to result in closed-form probability expressions for any and all expenditure patterns. However, the derivation is tedious, and the expressions get unwieldy. Pinjari and Bhat (2008) formulate a special two-level nested case of the MDCGEV model with a nested extreme value distributed structure that has the following joint cumulative distribution:

$$F(\varepsilon_1, \varepsilon_2, .., \varepsilon_K) = \exp\left[-\sum_{s=1}^{S_K} \left\{ \sum_{i \in s^{\text{th}} \text{nest}} \exp\left(-\frac{\varepsilon_i}{\theta_s} \right) \right\}^{\theta_s} \right] \tag{21}$$

In the above expression, $s\,(=1,2,...,S_K)$ is the index to represent a nest of alternatives, S_K is the total number of nests the *K* alternatives belong to, and $\theta_s (0 < \theta_s \leq 1; s = 1,2,...,S_K)$ is the (dis)similarity parameter introduced to induce correlations among the stochastic components of the utilities of alternatives belonging to the s^{th} nest.[12]

Without loss of generality, let $1,2,...,S_M$ be the nests the *M* chosen alternatives belong to, and let $q_1, q_2, ..., q_{S_M}$ be the number of chosen alternatives in each of the S_M nests (thus, $q_1 + q_2 + ... + q_{S_M} = M$). Using the nested extreme value error distribution assumption specified in Equation (21) (and the above-identified notation), Pinjari and Bhat (2008) derived the following expression for the multiple discrete-continuous nested extreme value (MDCNEV) model:

[12] This error structure assumes that the nests are mutually exclusive and exhaustive (*i.e.*, each alternative can belong to only one nest and all alternatives are allocated to one of the S_K nests).

$$P(t_1^*, t_2^*, \ldots t_M^*,\ 0, \ldots, 0) =$$

$$|J| \frac{\prod_{i \in \{\text{chosen alts}\}} e^{\frac{V_i}{\theta_i}}}{\prod_{s=1}^{S_M} \left(\sum_{i \in s^{\text{th}} \text{nest}} e^{\frac{V_i}{\theta_s}} \right)^{q_s}} \sum_{r_1=1}^{q_1} \ldots \sum_{r_s=1}^{q_s} \ldots \sum_{r_{S_M}=1}^{q_{S_M}} \left\{ \prod_{s=1}^{S_M} \left[\frac{\left(\sum_{i \in s^{\text{th}} \text{nest}} e^{\frac{V_i}{\theta_s}} \right)^{\theta_s}}{\sum_{s=1}^{S_k} \left\{ \left(\sum_{i \in s^{\text{th}} \text{nest}} e^{\frac{V_i}{\theta_s}} \right)^{\theta_s} \right\}} \right]^{q_s - r_s + 1} \left(\prod_{s=1}^{S_M} sum(X_{rs}) \right) \left(\sum_{s=1}^{S_M} (q_s - r_s + 1) - 1 \right)! \right\} \quad (22)$$

In the above expression, $sum(X_{rs})$ is the sum of elements of a row matrix X_{rs} (see Appendix A for a description of the form of the matrix X_{rs}).

As indicated in Pinjari and Bhat (2008), the general expression above represents the MDCNEV consumption probability for any consumption pattern with a two-level nested extreme value error structure. It may be verified that the MDCNEV probability expression in Equation (22) simplifies to Bhat's (2008) MDCEV probability expression when each of the utility functions are independent of one another (*i.e.*, when $\theta_s = 1$ and $q_s = 1 \; \forall s$, and $S_M = M$).

4.3 The Mixed MDCEV Model

The MDCGEV structure is able to accommodate flexible correlation patterns. However, it is unable to accommodate random taste variation, and it imposes the restriction of equal scale of the error terms. Incorporating a more general error structure is straightforward through the use of a mixing distribution, which leads to the Mixed MDCEV (or MMDCEV) model. Specifically, the error term, ε_k, may be partitioned into two components, ζ_k and η_k. The first component, ζ_k, can be assumed to be independently and identically Gumbel distributed across alternatives with a scale parameter of σ. The second component, η_k, can be allowed to be correlated across alternatives and to have a heteroscedastic scale. Let $\eta = (\eta_1, \eta_2, \ldots, \eta_K)'$, and assume that η is distributed multivariate normal, $\eta \sim N(0, \Omega)$.[13]

[13] Other distributions may also be used for η. Note that the distribution of η can arise from an error components structure or a random coefficients structure or a combination of the two, similar to the case of the usual mixed logit model (see Bhat, 2007).

For given values of the vector η, one can follow the discussion of the earlier section and obtain the usual MDCEV probability that the first M of the k goods are consumed. The unconditional probability can then be computed as:

$$P\left(e_1^*, e_2^*, e_3^*, ..., e_M^*, 0, 0, ..., 0\right)$$

$$= \int_{\eta} \frac{1}{\sigma^{M-1}} \left[\prod_{i=1}^{M} c_i\right]\left[\sum_{i=1}^{M} \frac{1}{c_i}\right]\left[\frac{\prod_{i=1}^{M} e^{(V_i+\eta_i)/\sigma}}{\left(\sum_{k=1}^{K} e^{(V_k+\eta_k)/\sigma}\right)^M}\right](M-1)!\, dF(\eta). \tag{23}$$

where F is the multivariate cumulative normal distribution (see Bhat, 2005; Bhat and Sen, 2006; and Bhat *et al.*, 2006).

The model in Equation (23) can be extended in a conceptually straightforward manner to also include random coefficients on the independent variables z_k, and random-effects (or even random coefficients) in the α_k satiation parameters (if the α profile is used) or the γ_k parameters (if the γ profile is used).

4.3.1 Heteroscedastic structure within the MMDCEV framework

Consider the case where there is price variation across the alternatives, and the overall errors ε_k are heteroscedastic, but not correlated. Assuming a 4-alternative case for ease in presentation, the heteroscedastic structure may be specified in the form of the following covariance matrix for $\varepsilon = (\varepsilon_{k1}, \varepsilon_{k2}, \varepsilon_{k3}, \varepsilon_{k4})$:

$$Cov(\varepsilon) = \frac{\pi^2 \sigma^2}{6}\begin{bmatrix} 1 & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{bmatrix} + \begin{bmatrix} \omega_1^2 & 0 & 0 & 0 \\ & \omega_2^2 & 0 & 0 \\ & & \omega_3^2 & 0 \\ & & & \omega_4^2 \end{bmatrix}, \tag{24}$$

where the first component on the right side corresponds to the IID covariance matrix of $\zeta = (\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ and the second component is the heteroscedastic covariance matrix of η. The covariance of error differences with respect to the first alternative is:

$$Cov(\tilde{\varepsilon}_1 = \tilde{\varepsilon}_{21}, \tilde{\varepsilon}_{31}, \tilde{\varepsilon}_{41}) = \frac{\pi^2\sigma^2}{6}\begin{bmatrix} 2 & 1 & 1 \\ & 2 & 1 \\ & & 2 \end{bmatrix} + \begin{bmatrix} \omega_1^2 + \omega_2^2 & \omega_1^2 & \omega_1^2 \\ & \omega_1^2 + \omega_3^2 & \omega_1^2 \\ & & \omega_1^2 + \omega_4^2 \end{bmatrix} \tag{25}$$

An inspection of the matrix above shows only four independent equations (the rank condition), implying that at most four parameters are estimable[14]. There are two ways to proceed with a normalization, as discussed below.

The first approach is to normalize σ and estimate the heteroscedastic covariance matrix of η (*i.e.*, ω_1, ω_2, ω_3, and ω_4). Assume that σ is normalized to $\tilde{\sigma}$, and let the corresponding values of ω_k be $\tilde{\omega}_k$ (k = 1, 2, 3, 4). Then, the following equalities should hold, based on Equation (25), for any normalization of σ to $\tilde{\sigma}$ ($q = \pi^2 / 6$ below):

$$\begin{aligned} &\omega_1^2 + q\sigma^2 = \tilde{\omega}_1^2 + q\tilde{\sigma}^2 \\ &\omega_1^2 + \omega_k^2 + 2q\sigma^2 = \tilde{\omega}_1^2 + \tilde{\omega}_k^2 + 2q\tilde{\sigma}^2 \quad (k = 2, 3, 4) \end{aligned} \tag{26}$$

The above equalities can be rewritten as:

$$\tilde{\omega}_k^2 = \omega_k^2 + q\sigma^2 - q\tilde{\sigma}^2 \quad (k = 1, 2, 3, 4) \tag{27}$$

The normalized variance terms $\tilde{\omega}_k^2$ must be greater than or equal to zero, which implies that the following conditions should hold:

$$\omega_k^2 + q\sigma^2 \geq q\tilde{\sigma}^2 \quad (k = 1, 2, 3, 4) \tag{28}$$

Intuitively, the above condition implies that the normalization on $\tilde{\sigma}$ must be set low enough so that the overall "true" variance of each error term $(= \omega_k^2 + q\sigma^2)$ is larger than $q\tilde{\sigma}^2$. For example, setting σ to 1 would be inappropriate if the "true" variance of one or more alternatives is less than $\pi^2 / 6$. Since the "true" variance is unknown, the best the analyst can do is to normalize σ to progressively smaller values and statistically examine the results.

The second approach is to normalize one of the ω_k terms instead of the σ term. In this case, from Equation (25), we can write:

[14] Strictly speaking, one can estimate all the five parameters (σ, ω_1, ω_2, ω_3, and ω_4) because of the difference in the extreme value distributions used for ζ_k and the normal distributions used for η_k (see Walker, 2002). However, the model will be near singular, and it is important to place the order/rank constraint.

$$q\tilde{\sigma}^2 = \omega_1^2 + q\sigma^2 - \tilde{\omega}_1^2 = \frac{1}{2}\left[\omega_1^2 + \omega_k^2 + 2q\sigma^2 - \tilde{\omega}_1^2 - \tilde{\omega}_k^2\right];\ k = 2, 3, 4. \tag{29}$$

After some manipulations, the above equation may be rewritten as:

$$\tilde{\omega}_k^2 = \omega_k^2 + \tilde{\omega}_1^2 - \omega_1^2;\ k = 2, 3, 4. \tag{30}$$

Next, imposing the condition that the normalized terms $\tilde{\omega}_k^2$ must be greater than or equal to zero implies the following:

$$\tilde{\omega}_1^2 \geq \omega_1^2 - \omega_k^2\ \ (k = 2, 3, 4). \tag{31}$$

The above condition is automatically satisfied as long as the first alternative is the minimum variance alternative. An associated convenient normalization is $\tilde{\omega}_1^2 = 0$, since the resulting model nests the MDCEV model. The minimum variance alternative can be determined by estimating an unidentified model with all the k ω_k terms, and identifying the alternative with the minimum variance (see Walker *et al.*, 2004, for an equivalent procedure for a heteroscedastic specification within the mixed multinomial logit model).

The above discussion assumes there is price variation across goods. In the case of no price variation, the scale σ is not identifiable. In this case, the easiest procedure is to normalize σ to 1 and the ω_k^2 value for the minimum variance alternative k to zero.

4.3.2 The general error covariance structure within the MMDCEV framework

Appropriate identification normalizations will have to placed on σ and the covariance matrix of η when the analyst is estimating an error-components structure to allow correlation in unobserved factors influencing the baseline utility of alternatives, since only a (K-1)x(K-1) covariance of error differences is identified. This can be accomplished by imposing a structure based on *a priori* beliefs or intuitive considerations. However, the analyst must ensure that the elements of the assumed restricted covariance structure can be recovered from the (K-1)x(K-1) covariance of error differences that is actually estimable.

In the most general error covariance structure, and when there is price variation, one way to achieve identification is the following: (1) Normalize the scale parameter σ to be a small value such that the variance of the minimum variance alternative exceeds $\pi^2\sigma^2/6$ (since this variance is not known, the analyst will have to experiment with alternative fixed σ values), (2) Normalize ω_k for the minimum variance alternative *k* to zero, and (3) Normalize all correlations of this minimum variance alternative with other alternatives to zero. Together, these normalizations leave only $K(K-1)/2$ parameters to be estimated, and are adequate for identification. In the case of no price variation, an additional restriction will have to be imposed. One approach would be to set $\sum_{k=2}^{K}\omega_k^2 = 1$ to set the scale in the covariance matrix of η.

4.4 The Joint MDCEV-Single Discrete Choice Model

The MDCEV model and its extensions discussed thus far are suited for the case when the alternatives are imperfect substitutes, as recognized by the use of a non-linear utility that accommodates a diminishing marginal utility as the consumption of any alternative increases. However, there are many instances where the real choice situation is characterized by a combination of imperfect and perfect substitutes (perfect substitutes correspond to the case where consumers prefer to select only one discrete alternative at any choice occasion; see Hanemann, 1984). The MDCEV model needs to be modified to handle such a combination of a multiple discrete-continuous choice among alternatives, as well as a single choice of one sub-alternative within one or more of the alternatives. We do not discuss this case here due to space constraints, but the reader is referred to Bhat *et al.* (2009) and Bhat *et al.* (2006).

4.5. The Non-Additive MDCEV Model Structure

Vasquez and Hanemann (2008) have recently proposed an extension of Bhat's additively separable linear Box-Cox utility functional form (Equation 1) to incorporate a non-additively separable quadratic Box-Cox functional form. Using more flexible non-additive utility structures allows the analyst to handle both complementarity as well as substitution among goods. To write this general non-additive form, define μ_k as:

$$\mu_k = \sum_{k=1}^{K} \frac{\gamma_k}{\alpha_k}\left\{\left(\frac{x_k}{\gamma_k}+1\right)^{\alpha_k} - 1\right\}.$$

Then, a non-additively separable functional form may be written as:

$$U(\boldsymbol{x}) = \sum_{k=1}^{K} \mu_k \psi_k + \frac{1}{2} \sum_{k=1}^{K} \sum_{m=1}^{K} \theta_{km} \mu_k \mu_m .$$

This is very general, and collapses to Bhat's additively separable form when $\theta_{km} = 0$ for all k and m. It collapses to the translog functional form when $\alpha_k \to 0$ for all *k*, and to Wales and Woodland's quadratic form when $\alpha_k = 1$ for all *k*. The interpretation of the parameters is not as straightforward as in Bhat's MDCEV and the probability expressions for the consumption of the goods and the Jacobian do not have simple forms. But the gain is that the marginal utility of consumption of a good is not only dependent on the amount of that good consumed, but also the amount of other goods consumed.

5. THE MODEL WITH AN OUTSIDE GOOD

Thus far, the discussion has assumed that there is no outside numeraire good (*i.e.*, no essential Hicksian composite good). If an outside good is present, label it as the first good which now has a unit price of one. Also, for identification, let $\psi(x_1, \varepsilon_1) = e^{\varepsilon_1}$. Then, the utility functional form needs to be modified as follows:

$$U(\boldsymbol{x}) = \frac{1}{\alpha_1} \exp(\varepsilon_1) \left\{ (x_1 + \gamma_1)^{\alpha_1} \right\} + \sum_{k=2}^{K} \frac{\gamma_k}{\alpha_k} \exp(\beta' z_k + \varepsilon_k) \left\{ \left(\frac{x_k}{\gamma_k} + 1 \right)^{\alpha_k} - 1 \right\} \tag{32}$$

In the above formula, we need $\gamma_1 \leq 0$, while $\gamma_k > 0$ for k > 1. Also, we need $x_1 + \gamma_1 > 0$. The magnitude of γ_1 may be interpreted as the required lower bound (or a "subsistence value") for consumption of the outside good. As in the "no-outside good" case, the analyst will generally not be able to estimate both α_k and γ_k for the outside and inside goods. The analyst can estimate one of the following five utility forms:

$$U(\boldsymbol{x}) = \frac{1}{\alpha_1} \exp(\varepsilon_1) x_1^{\alpha_1} + \sum_{k=2}^{K} \frac{1}{\alpha_k} \exp(\beta' z_k + \varepsilon_k) \left\{ (x_k + 1)^{\alpha_k} - 1 \right\}$$

$$U(\boldsymbol{x}) = \frac{1}{\alpha_1} \exp(\varepsilon_1) x_1^{\alpha_1} + \sum_{k=2}^{K} \gamma_k \exp(\beta' z_k + \varepsilon_k) \ln\left(\frac{x_k}{\gamma_k} + 1 \right) \tag{33}$$

$$U(\boldsymbol{x}) = \frac{1}{\alpha}\exp(\varepsilon_1)x_1^{\alpha} + \sum_{k=2}^{K} \frac{\gamma_k}{\alpha}\exp(\beta' z_k + \varepsilon_k)\left\{\left(\frac{x_k}{\gamma_k}+1\right)^{\alpha} - 1\right\}$$

$$U(\boldsymbol{x}) = \exp(\varepsilon_1)\ln\{x_1 + \gamma_1\} + \sum_{k=2}^{K} \frac{1}{\alpha_k}\exp(\beta' z_k + \varepsilon_k)\left\{(x_k + 1)^{\alpha_k} - 1\right\}$$

$$U(\boldsymbol{x}) = \exp(\varepsilon_1)\ln\{x_1 + \gamma_1\} + \sum_{k=2}^{K} \gamma_k \exp(\beta' z_k + \varepsilon_k)\ln\left(\frac{x_k}{\gamma_k}+1\right)$$

The third functional form above is estimable because the constant α parameter is obtaining a "pinning effect" from the satiation parameter for the outside good. The analyst can estimate all the five possible functional forms and select the one that fits the data best based on statistical and intuitive considerations. The identification considerations discussed for the "no-outside good" case carries over to the "with outside good" case. The probability expression for the expenditure allocation on the various goods (with the first good being the outside good) is identical to Equation (19), while the probability expression for consumption of the goods (with the first good being the outside good) is

$$P\left(x_1^*, x_2^*, x_3^*, ..., x_M^*, 0, 0, ..., 0\right)$$
$$= \frac{1}{\sigma^{M-1}}\left[\prod_{i=1}^{M} f_i\right]\left[\sum_{i=1}^{M} \frac{p_i}{f_i}\right]\left[\frac{\prod_{i=1}^{M} e^{V_i/\sigma}}{\left(\sum_{k=1}^{K} e^{V_k/\sigma}\right)^{M}}\right](M-1)!, \tag{34}$$

where $f_i = \left(\dfrac{1-\alpha_i}{x_i^* + \gamma_i}\right)$.

The expressions for V in Equation (19) and Equation (34) are as follows for each of the five utility forms in Equation (33):

First form - $V_k = \beta' z_k + (\alpha_k - 1)\ln(x_k^* + 1) - \ln p_k \;\; (k \ge 2);\; V_1 = (\alpha_1 - 1)\ln(x_1^*)$

Second form - $V_k = \beta' z_k - \ln(\dfrac{x_k^*}{\gamma_k} + 1) - \ln p_k \;\; (k \ge 2);\; V_1 = (\alpha_1 - 1)\ln(x_1^*)$ (35)

Third form - $V_k = \beta' z_k + (\alpha - 1)\ln(\dfrac{x_k^*}{\gamma_k} + 1) - \ln p_k \;\; (k \ge 2);\; V_1 = (\alpha - 1)\ln(x_1^*)$

Fourth form - $V_k = \beta' z_k + (\alpha_k - 1)\ln(x_k^* + 1) - \ln p_k \;\; (k \geq 2);\; V_1 = -\ln(x_1^* + \gamma_1)$

Fifth form - $V_k = \beta' z_k - \ln(\frac{x_k^*}{\gamma_k} + 1) - \ln p_k \;\; (k \geq 2);\; V_1 = -\ln(x_1^* + \gamma_1)$

6. APPLICATIONS

The MDCEV model framework has been employed in modeling a number of choice situations that are characterized by multiple-discreteness. These can be broadly categorized into the following research areas: (1) activity time-use analysis (adults and children), (2) household vehicle ownership, (3) household expenditures and (4) Angler's site choice.[15]

6.1 Activity Time-Use Analysis

The MDCEV model that assumes diminishing marginal utility of consumption provides an ideal platform for modeling activity time-use decisions. The different studies on activity time-use are described chronologically below.

Bhat (2005) demonstrated an application of the MDCEV model to individual time use in different types of discretionary activity pursuits on weekend days. The modeling exercise included different kinds of variables, including household demographics, household location variables, individual demographics and employment characteristics, and day of week and season of year. Bhat *et al.* (2006) formulate a unified utility-maximizing framework for the analysis of a joint imperfect-perfect substitute goods case. This is achieved by using a satiation-based utility structure (MDCEV) across the imperfect substitutes, but a simple standard discrete choice-based linear utility structure (MNL) within perfect substitutes. The joint model is applied to analyze individual time-use in both maintenance and leisure activities using weekend day time-use.

Kapur and Bhat (2007) specifically modeled the social context of activity participation by examining the accompaniment arrangement (i.e., company type) in activity participation. Sener and Bhat (2007) also examined participation and time investment in in-home leisure as well as out-of-home discretionary activities with a specific emphasis on the accompanying individuals in

[15] The summary of all the studies discussed in this chapter are compiled in the form of a table with information on the application focus, the data source used for the empirical analysis, the number and labels of discrete alternatives, the continuous component in the empirical context and the MDCEV model type employed. The table is available to the readers at: http://www.caee.utexas.edu/prof/bhat/ABSTRACTS/MDCEV_BookChapter_Table1.pdf

children's activity engagement. Copperman and Bhat (2007) formulated a comprehensive framework to consider participation, and levels of participation, in physically passive and physically active episodes among children on weekend days.

LaMondia *et al.* (2008) focused their attention on vacation travel in USA. Specifically, the paper examined how households decide what vacation travel activities to participate in on an annual basis, and to what extent, given the total annual vacation travel time that is available at their disposal.

The models presented in Sener *et al.* (2008) offer a rich framework for categorizing and representing the activity-travel patterns of children within larger travel demand model systems. The paper provides a taxonomy of child activities that explicitly considers the spatial and temporal constraints that may be associated with different types of activities.

Pinjari *et al.* (2009) presented a joint model system of residential location and activity time-use choices. The model system takes the form of a joint mixed Multinomial Logit–Multiple Discrete-Continuous Extreme Value (MNL–MDCEV) structure that (a) accommodates differential sensitivity to the activity-travel environment attributes due to both observed and unobserved individual-related attributes, and (b) controls for the self selection of individuals into neighborhoods due to both observed and unobserved individual-related factors.

Spissu *et al.* (2009) formulated a panel version of the Mixed Multiple Discrete Continuous Extreme Value (MMDCEV) model that is capable of simultaneously accounting for repeated observations from the same individuals (panel), participation in multiple activities in a week, durations of activity engagement in various activity categories, and unobserved individual-specific factors affecting discretionary activity engagement including those common across pairs of activity category utilities.

Pinjari and Bhat (2008) proposed the MDCNEV model that captures inter-alternative correlations among alternatives in mutually exclusive subsets (or nests) of the choice set, while maintaining the closed-form of probability expressions for any (and all) consumption pattern(s). The model estimation results provide several insights into the determinants of non-workers' activity time-use and timing decisions.

Rajagopalan *et al.* (2009) predicted workers' activity participation and time allocation patterns in seven types of out-of-home non-work activities at various time periods of the day. The knowledge of the activities (and the corresponding time allocations and timing decisions) predicted by this model can be used for subsequent detailed scheduling and sequencing of activities and related travel in an activity-based microsimulation framework.

6.2 Household Vehicle Ownership

The MDCEV framework, with its capability to handle multiple-discreteness, lends itself very well to model household vehicle ownership by type.

Bhat and Sen (2006) modeled the simultaneous holdings of multiple vehicle types (passenger car, SUV, pickup truck, minivan and van), as well as determined the continuous miles of usage of each vehicle type. The model can be used to determine the change in vehicle type holdings and usage due to changes in independent variables over time. As a demonstration, the impact of an increase in vehicle operating costs, on vehicle type ownership and usage, is assessed. Ahn *et al.* (2008) employed conjoint analysis and the MDCEV framework to understand consumer preferences for alternative fuel vehicles. The results indicate a clear preference of gasoline-powered cars among consumers, but alternative fuel vehicles offer a promising substitute to consumers. Bhat *et al.* (2009) formulated and estimated a nested model structure that includes a multiple discrete-continuous extreme value (MDCEV) component to analyze the choice of vehicle type/vintage and usage in the upper level and a multinomial logit (MNL) component to analyze the choice of vehicle make/model in the lower nest.

6.3 Household Expenditures

The MDCEV framework provides a feasible framework to analyze consumption patterns. Ferdous *et al.* (2008) employed a MDCNEV structure to explicitly recognize that people choose to consume multiple goods and commodities. Model results show that a range of household socio-economic and demographic characteristics affect the percent of income or budget allocated to various consumption categories and savings. Rajagopalan and Srinivasan (2008) explicitly investigated transportation related household expenditures by mode. Specifically, they examined the mode choice and modal expenditures at the household level. The model results indicate that mode choice and frequency decisions are influenced by prior mode choice decisions, and the user’s perception of safety and congestion.

6.4 Angler's Site Choice

Vasquez and Hanemann (2008) formulate the non-additive MDCEV model structure to study angler site choice. In this study, they employ individual level variables such as skill, leisure time available, and ownership status (of cabin, boat or RV). Further, they undertake the computation of welfare measures using a sequential quadratic programming method.

7. CONCLUSIONS

Classical discrete and discrete-continuous models deal with situations where only one alternative is chosen from a set of mutually exclusive alternatives. Such models assume that the alternatives are perfectly substitutable for each other. On the other hand, many consumer choice situations are characterized by the simultaneous demand for multiple alternatives that are imperfect substitutes or even complements for one another.

This book chapter discusses the multiple discrete-continuous extreme value (MDCEV) model and its many variants. Recent applications of the MDCEV type of models are presented and briefly discussed. This overview of applications indicates that the MDCEV model has been employed in many different empirical contexts in the transportation field, and also highlights the potential for application of the model in several other fields. The overview also serves to highlight the fact that the field is at an exciting and ripe stage for further applications of the multiple discrete-continuous models. At the same time, several challenges lie ahead, including (1) Accommodating more than one constraint in the utility maximization problem (for example, recognizing both time and money constraints in activity type choice and duration models; see Anas, 2006 for a recent theoretical effort to accommodate such multiple constraints), (2) Incorporating latent consideration sets in a theoretically appropriate way within the MDCEV structure (the authors are currently addressing this issue in ongoing research), (3) Using more flexible utility structures that can handle both complementarity as well as substitution among goods, and that do not impose the constraints of additive separability (Vasquez and Hanemann, 2008 provide some possible ways to accommodate this), and (4) Developing easy-to-apply techniques to use the model in forecasting mode.

REFERENCES

Ahn, J., Jeong, G., Kim, Y. 2008 A forecast of household ownership and use of alternative fuel vehicles: A multiple discrete-continuous choice approach. *Energy Economics*, 30 (5), 2091-2104.

Amemiya, T., 1974. Multivariate regression and simultaneous equation models when the dependent variables are truncated normal. *Econometrica*, 42, 999-1012.

Anas, A., 2006. A unified theory of consumption and travel. Presented at the Conference Honoring Kenneth A. Small, Department of Economics, University of California, Irvine, February 3-4.

Arora, N., Allenby, G.M., and Ginter, J.L., 1998. A hierarchical Bayes model of primary and secondary demand. *Marketing Science*, 17, 29-44.

Baltas, G., 2004. A model for multiple brand choice. *European Journal of Operational Research*, 154, 144-149.

Bhat, C.R., 2005. A multiple discrete-continuous extreme value model: formulation and application to discretionary time-use decisions. *Transportation Research Part B*, 39(8), 679-707.

Bhat, C.R., 2007. Econometric choice formulations: alternative model structures, estimation techniques, and emerging directions. In *Moving Through Nets: The Physical and Social Dimensions of Travel - Selected papers from the 10th International Conference on Travel Behaviour Research*, K. W. Axhausen (Ed.), Elsevier, pp. 45-80.

Bhat, C.R., 2008. The Multiple Discrete-Continuous Extreme Value (MDCEV) Model: Role of Utility Function Parameters, Identification Considerations, and Model Extensions. *Transportation Research Part B*, 42(3), 274-303.

Bhat, C.R., and Sen, S., 2006. Household vehicle type holdings and usage: an application of the multiple discrete-continuous extreme value (MDCEV) model. *Transportation Research Part B*, 40(1), 35-53.

Bhat, C.R., and Srinivasan, S., 2005. A multidimensional mixed ordered-response model for analyzing weekend activity participation. *Transportation Research Part B*, 39(3), 255-278.

Bhat, C.R., Srinivasan, S., and Sen, S., 2006. A joint model for the perfect and imperfect substitute goods case: application to activity time-use decisions. *Transportation Research Part B*, 40(10), 827-850.

Bhat, C.R., S. Sen, and N. Eluru, 2009. The Impact of Demographics, Built Environment Attributes, Vehicle Characteristics, and Gasoline Prices on Household Vehicle Holdings and Use. *Transportation Research Part B*, 43(1), 1-18.

Chan, T.Y., 2006. Estimating a continuous hedonic-choice model with an application to demand for soft drinks. *RAND Journal of Economics*, 37(2), 466-482, Summer.

Chiang, J., 1991. A simultaneous approach to whether to buy, what to buy, and how much to buy. *Marketing Science*, 4, 297-314.

Chintagunta, P.K., 1993. Investigating purchase incidence, brand choice and purchase quantity decisions of households. *Marketing Science*, 12, 194-208.

Copperman, R., and C.R. Bhat, 2007.An Analysis of the Determinants of Children's Weekend Physical Activity Participation. *Transportation*, 34(1), 67-87.

Deaton, A., and Muellbauer, J., 1980. *Economics and Consumer Behavior*. Cambridge University Press, Cambridge.

Dube, J.P., 2004. Multiple discreteness and product differentiation: Demand for carbonated soft drinks. *Marketing Science*, 23(1), 66-81.

Edwards, Y.D., and Allenby, G.M., 2003. Multivariate analysis of multiple response data. *Journal of Marketing Research*, 40, 321-334.

Ferdous, N., A.R. Pinjari, C.R. Bhat, and R.M. Pendyala, 2008. A comprehensive analysis of household transportation expenditures relative to other goods and services: An application to United States consumer expenditure data, Technical paper, Department of Civil, Architectural & Environmental Engineering, The University of Texas at Austin, July.

Hanemann, W.M., 1978. A methodological and empirical study of the recreation benefits from water quality improvement. Ph.D. dissertation, Department of Economics, Harvard University.

Hanemann, W.M., 1984. The discrete/continuous model of consumer demand. *Econometrica*, 52, 541-561.

Hendel, I., 1999. Estimating multiple-discrete choice models: An application to computerization returns. *Review of Economic Studies*, 66, 423-446.

Herriges, J.A., Kling, C.L., and Phaneuf, D.J., 2004. What's the use? Welfare estimates from revealed preference models when weak complementarity does not hold. *Journal of Environmental Economics and Management*, 47, 55-70.

Kapur, A., and C.R. Bhat, 2007. On Modeling Adults' Daily Time Use by Activity Purpose and Accompaniment Arrangement. *Transportation Research Record*, 2021, 18-27.

Kim, J., Allenby, G.M., and Rossi. P.E., 2002. Modeling consumer demand for variety. *Marketing Science*, 21, 229-250.

Kuhn, H.W. and Tucker, A.W., 1951. Nonlinear programming. In *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, J. Neyman (Ed.), University of California Press, Berkeley, CA, 481-492.

LaMondia, J., C.R. Bhat, and D.A. Hensher, 2008. An Annual Time Use Model for Domestic Vacation Travel. *Journal of Choice Modelling*, 1(1), 70-97.

Mäler, K-G., 1974. *Environmental Economics: A Theoretical Inquiry*. The Johns Hopkins University Press for Resources for the Future, Baltimore, MD.

Manchanda, P., Ansari, A., and Gupta, S., 1999. The "shopping basket": A model for multicategory purchase incidence decisions. *Marketing Science*, 18, 95-114.

Mohn, C. and M. Hanemann (2005) Caught in a corner: using the Kuhn-Tucker conditions to value Montana sportfishing. In *Explorations in Environmental and Natural Resource Economics: Essays in Honor of Gardner M. Brown, Jr*., Halvorsen, R. and Layton, D.F. (eds.), Ch 10, pp. 188-207, Edward Elgar Publishing, Inc., Northampton, MA.

Phaneuf, D.J., and Smith, V.K., 2005. Recreation demand models. In *Handbook of Environmental Economics*, Volume 2, K-G. Mäler and J.R. Vincent (Eds.), North Holland.

Phaneuf, D.J., Kling, C.L., and Herriges, J.A., 2000. Estimation and welfare calculations in a generalized corner solution model with an application to recreation demand. *The Review of Economics and Statistics*, 82(1), 83-92.

Pinjari, A.R., and C.R. Bhat, 2008. A Multiple Discrete-Continuous Nested Extreme Value (MDCNEV) Model: Formulation and Application to Non-Worker Activity Time-Use and Timing Behavior on Weekdays. In press, *Transportation Research Part B*.

Pinjari, A.R., C.R. Bhat, and D.A. Hensher, 2009. Residential self-selection effects in an activity time-use behavior model. *Transportation Research Part B*, 43(7), 729-748.

Pollak, R., and Wales, T., 1992. *Demand System Specification and Estimation*. Oxford University Press, New York.

Rajagopalan, B.S., and K.K. Srinivasan (2008) Integrating Household-Level Mode Choice and Modal Expenditure Decisions in a Developing Country: Multiple Discrete-Continuous Extreme Value Model. Transportation Research Record: Journal of the Transportation Research Board, No. 2076, Transportation Research Board of the National Academies, Washington, D.C., 2008, pp. 41–51.

Rajagopalan, B.S., A.R. Pinjari, and C.R. Bhat, 2009. A Comprehensive Model of Workers' Non-Work Activity Time-Use and Timing Behavior. Forthcoming, *Transportation Research Record.*

Sener, I.N., and C.R. Bhat, 2007. An Analysis of the Social Context of Children's Weekend Discretionary Activity Participation. *Transportation*, 34(6), 697-7217.

Sener, I.N., R.B. Copperman, R.M. Pendyala, and C.R. Bhat, 2008. An Analysis of Children's Leisure Activity Engagement: Examining the Day of Week, Location, Physical Activity Level, and Fixity Dimensions. *Transportation*, 35(5), 673-696.

Spissu, E., A.R. Pinjari, C.R. Bhat, R.M. Pendyala, and K.W. Axhausen, 2009. An analysis of weekly out-of-home discretionary activity participation and time-use behavior. *Transportation*, 36(5), 483-510.

Tobin, J., 1958. Estimation of relationships for limited dependent variables. *Econometrica*, 26, 24-36.

Train, K., 2003. *Discrete Choice Methods with Simulation*. Cambridge University Press, Cambridge, United Kingdom.

Vasquez Lavin, F., and W.M. Hanemann, 2008. Functional Forms in Discrete/Continuous Choice Models With General Corner Solution. UC Berkeley, Department of Agricultural and Resource Economics, UCB.

von Haefen, R.H., 2003. Latent consideration sets and continuous demand system models. Working paper, Department of Agricultural & Resource Economics, University of Arizona.

von Haefen, R.H., 2004. Empirical strategies for incorporating weak complementarity into continuous demand system models. Working paper, Department of Agricultural and Resource Economics, University of Arizona.

von Haefen, R.H., and Phaneuf, D.J., 2005. Kuhn-Tucker demand system approaches to nonmarket valuation. In *Applications of Simulation Methods in Environmental and Resource Economics*, R. Scarpa and A.A. Alberini (Eds.), Springer.

von Haefen, R.H., Phaneuf, D.J., and Parsons, G.R., 2004. Estimation and welfare analysis with large demand systems. *Journal of Business and Economic Statistics*, 22(2), 194-205

Wales, T.J., and Woodland, A.D., 1983. Estimation of consumer demand systems with binding non-negativity constraints. *Journal of Econometrics*, 21(3), 263-85.

Walker, J., 2002. Mixed logit (or logit kernel) model: Dispelling misconceptions of identification. *Transportation Research Record*, 1805, 86-98.

Walker, J., Ben-Akiva, M., and Bolduc, D., 2004. Identification of the logit kernel (or mixed logit) model. Presented at the 10^{th} International Conference on Travel Behavior Research, Lucerne, Switzerland, August.

Walsh, J.W., 1995. Flexibility in consumer purchasing for uncertain future tastes. *Marketing Science*, 14, 148-165.

Appendix A

For $r_s = 1$, $X_{rs} = \{1\}$.

For $r_s = 2$, $X_{rs} = \left\{ \frac{(q_s - 1)(1-\theta_s)}{\theta_s} + \frac{(q_s - 2)(1-\theta_s)}{\theta_s} + \ldots + \frac{2(1-\theta_s)}{\theta_s} + \frac{1(1-\theta_s)}{\theta_s} \right\}$.

For $r_s = 3,4,\ldots,q_s$, X_{rs} is a matrix of size $\begin{bmatrix} q_s - 2 \\ r - 2 \end{bmatrix}$ which is formed as described below:

Consider the following row matrices A_{q_s} and A_{rs} (with the elements arranged in the descending order, and of size $q_s - 1$ and $r_s - 2$, respectively):

$$A_{q_s} = \left\{ \frac{(q_s - 1)(1-\theta_s)}{\theta_s}, \frac{(q_s - 2)(1-\theta_s)}{\theta_s}, \frac{(q_s - 3)(1-\theta_s)}{\theta_s}, \ldots, \frac{3(1-\theta_s)}{\theta_s}, \frac{2(1-\theta_s)}{\theta_s}, \frac{1(1-\theta_s)}{\theta_s} \right\}$$

$$A_{rs} = \left\{ r_s - 2, r_s - 3, r_s - 4, \ldots, 3, 2, 1 \right\}.$$

Choose any $r_s - 2$ elements (other than the last element, $\frac{1-\theta_s}{\theta_s}$) of the matrix A_{q_s} and arrange them in the descending order into another matrix A_{iq_s}. Note that we can form $\begin{bmatrix} q_s - 2 \\ r_s - 2 \end{bmatrix}$ number of such matrices. Subsequently, form another matrix $A_{irq_s} = A_{iq_s} + A_{rs}$. Of the remaining elements in the A_{q_s} matrix, discard the elements that are larger than or equal to the smallest element of the A_{iq_s} matrix, and store the remaining elements into another matrix labeled B_{irq_s}. Now, an element of X_{rs} (*i.e., x_{irq_s}*) is formed by performing the following operation:

$x_{irq_s} = \text{Product}(A_{irq_s}) \times \text{Sum}(B_{irq_s})$; that is, by multiplying the product of all elements of the matrix A_{irq_s} with the sum of all elements of the matrix B_{irq_s}. Note that the number of such elements of the matrix X_{rs} is equal to $\begin{bmatrix} q_s - 2 \\ r_s - 2 \end{bmatrix}$.

LIST OF FIGURES

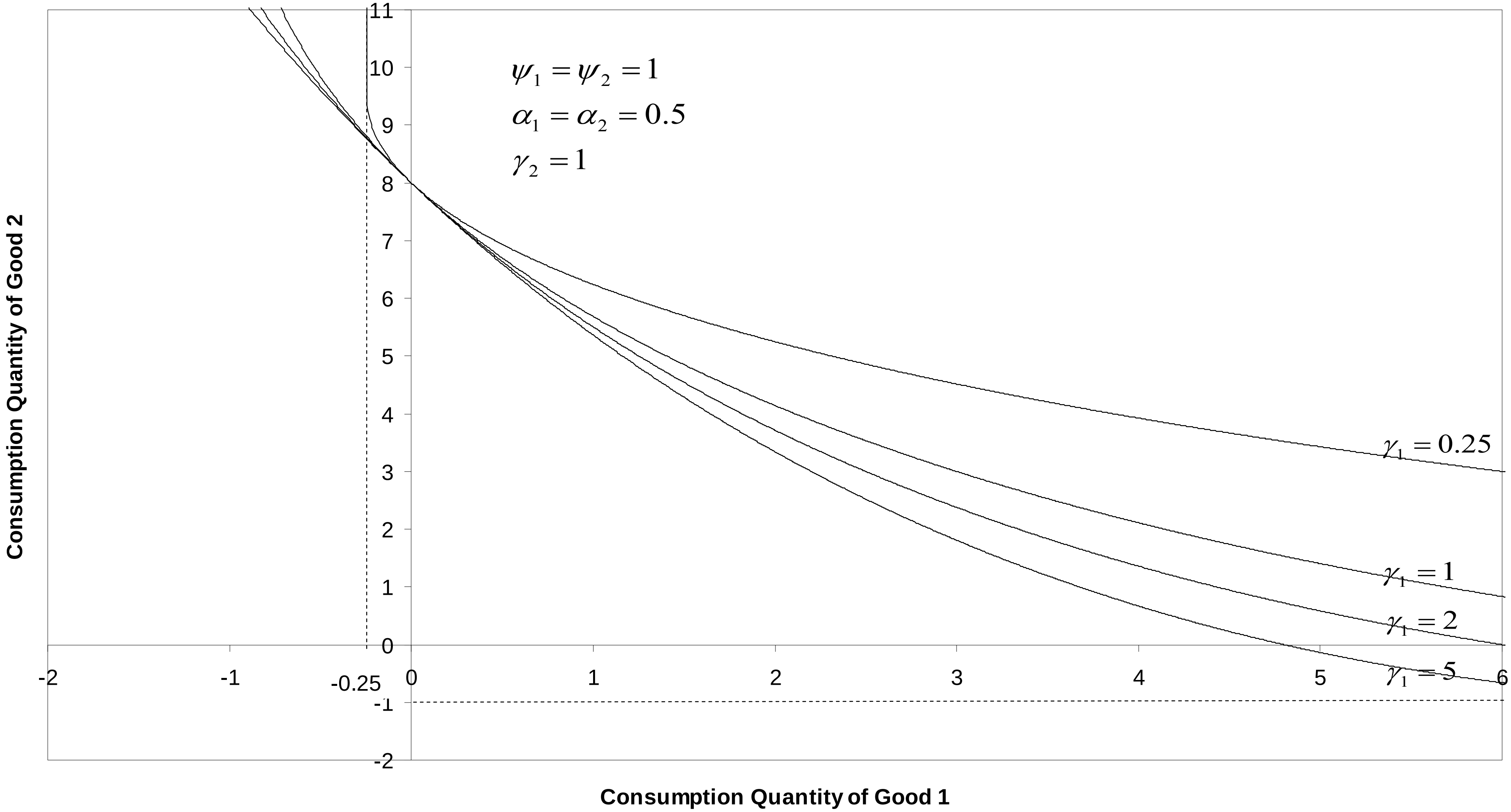

Figure 1. Indifference Curves Corresponding to Different Values of γ_1

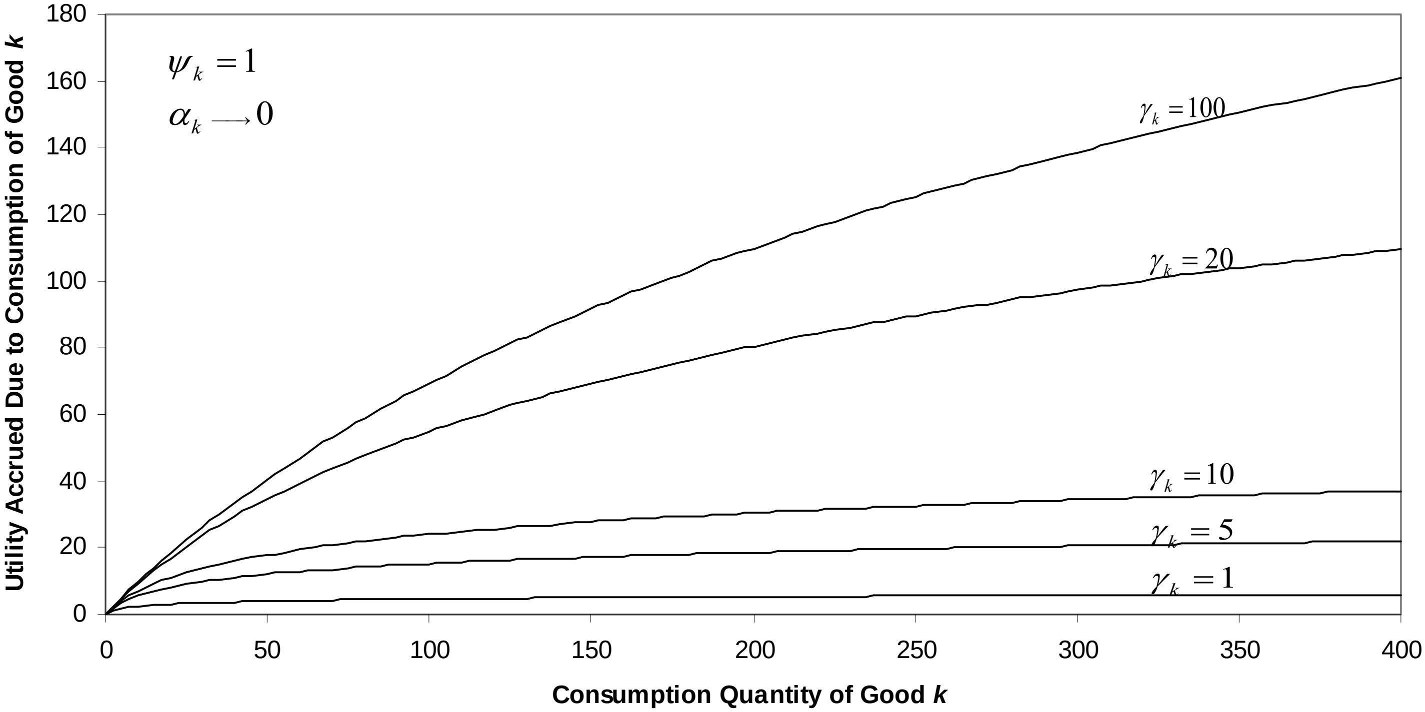

Figure 2. Effect of γ_k Value on Good *k*'s Subutility Function Profile

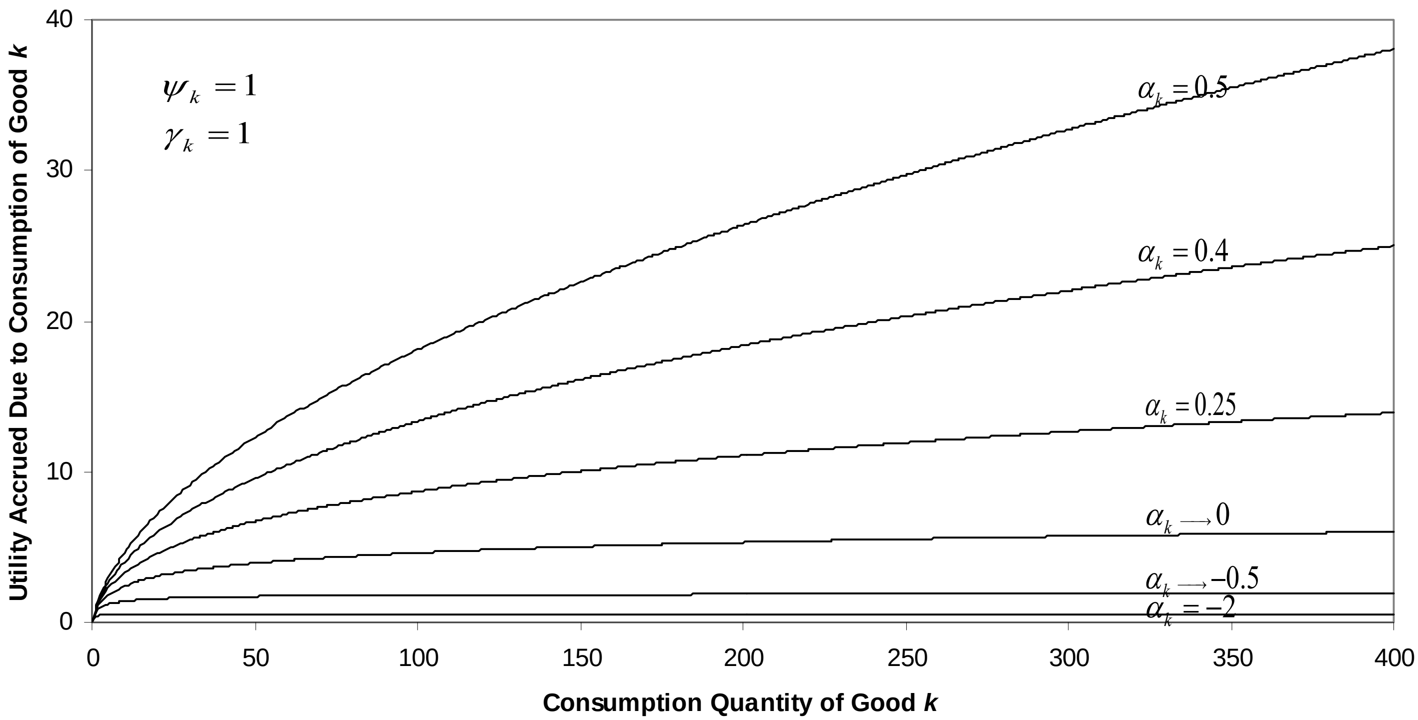

Figure 3. Effect of α_k Value on Good *k*'s Subutility Function Profile